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Free energy amplitude in finite-size scaling: the Baxter model

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Abstract. The critical amplitude A_0 of the free energy density of the Baxter model is studied numerically on strips with widths $L = 2$-12 as a function of the four-spin interaction K_4. It remains almost constant, near to double the exactly known Ising result ($A_0 = 2A_0^{\text{Ising}} = \pi/6$) in the range of K_4 values for which an extrapolation to infinite width is possible. possible.

1. Introduction

According to finite-size scaling theory (Fisher 1971, Barber 1983), the critical value of the free energy density on a strip with width L and periodic boundary conditions varies with L in two dimensions like

$$f_0(L) = f^0 + A_0 L^{-2} \qquad L \to \infty \tag{1.1}$$

where f^0 is the bulk free energy density and A_0 is a critical amplitude which is expected to be universal (Privman and Fisher 1984).

Non-universal behaviour is known to occur when a marginal scaling field is present (Kadanoff and Wegner 1971). This is observed in the exactly solved symmetric eight-vertex model (Baxter 1972) where the temperature exponent

$$y_t = 2 - (2/\pi)\cos^{-1}\tanh(2K_4) \tag{1.2}$$

varies continuously with the four-spin interaction K_4 which enters the Ising formulation of this model whereas the magnetic exponent keeps its Ising value $y_h = \frac{15}{8}$ (Barber and Baxter 1973). In the Ising formulation, the Hamiltonian is

$$-\beta H = K_2 \sum_{(i,j)} s_i s_j + K_4 \sum_{(i,j,k,l)} s_i s_j s_k s_l \tag{1.3}$$

where the first sum is over next-nearest-neighbour bonds and the second over elementary plaquettes on a square lattice (figure 1). The critical line is given by

$$\sinh(2K_2) = \exp(-2K_4). \tag{1.4}$$

Recently (Nightingale and Blöte 1983), the inverse correlation length amplitudes a_t, a_m, a_p for the energy, magnetisation and polarisation correlations have been studied. The numerical results support a well known relation with the anomalous dimensions of the corresponding operators (Pichard and Sarma 1981, Luck 1982, Cardy 1984)

$$a_j = 2\pi x_j \qquad (j = \text{t, m, p}). \tag{1.5}$$

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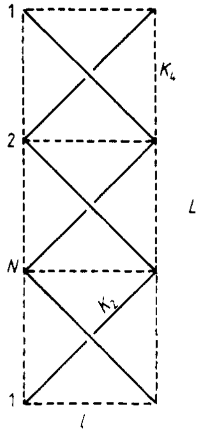

Figure 1. Transfer matrix used for the calculation of the free energy density $f_0(L)$. Periodic boundary conditions are used. K_2 is the second-neighbour interaction (bold lines) and K_4 the four-spin interaction (broken lines). When K_4 vanishes, the Baxter lattice is decoupled into two independent Ising sublattices.

The a_j are related to the appropriate free energy level amplitudes A_α (Privman and Fisher 1984, Turban and Debierre 1986) through

$$a_j = A_0 - A_\alpha . \tag{1.6}$$

The A_α may be deduced from the successive eigenvalues Λ_α of the transfer matrix at the critical point on an $L\times\infty$ cylinder built up of $L\times l$ slices

$$f_\alpha(L) = (1/lL)\ln\Lambda_\alpha = f^0 + A_\alpha L^{-2} \tag{1.7}$$

with $\Lambda_0 > \Lambda_1 \geqslant \Lambda_2 \geqslant \dots$.

In the present work, we study numerically the free energy amplitude A_0 of the Baxter model. Our purpose was to see whether A_0 remains universal in the presence of a marginal operator and if not, to find out an analytic expression for the K_4 dependence. Our numerical results are presented in § 2 and discussed in § 3.

2. Numerical results

The free energy density is obtained using the transfer matrix of figure 1 for $N = 2$–12. choosing l as unit length, L is equal to N, the number of spins in a column. The free energy density is also the free energy per site (figures 2 and 3) given by

$$f_0(N) = \ln\Lambda_0/N = f^0 + A_0(N)N^{-2} \tag{2.1}$$

$$A_0(N) = A_0(1 + bN^{-y}). \tag{2.2}$$

A power law correction to scaling has been included in $A_0(N)$. Even values of the width are taken in order to get two decoupled Ising sublattices when K_4 vanishes. $A_0(N)$ and $f^0(N)$ are estimated for odd N values using results for $N-1$ and $N+1$:

$$A_0(N) = \frac{f_0(N+1) - f_0(N-1)}{(N+1)^{-2} - (N-1)^{-2}} \simeq A_0(1 + cN^{-y}) \tag{2.3}$$

$$f^0(N) = \frac{(N+1)^2 f_0(N+1) - (N-1)^2 f_0(N-1)}{(N+1)^2 - (N-1)^2} \simeq f^0(1 + dN^{-y-2}). \tag{2.4}$$

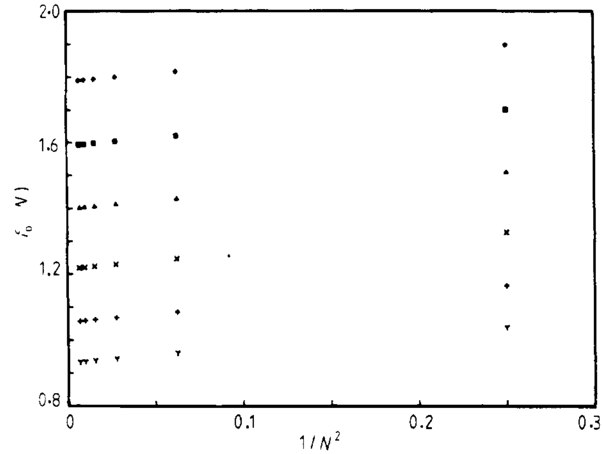

Figure 2. Free energy density of the Baxter model as a function of $1/N^2$ where N is the number of spins in a column, for negative values of the four-spin interaction K_4 (◆, −1.0; ■, −0.8; ▲, −0.6; ×, −0.4; +, −0.2; Y, 0).

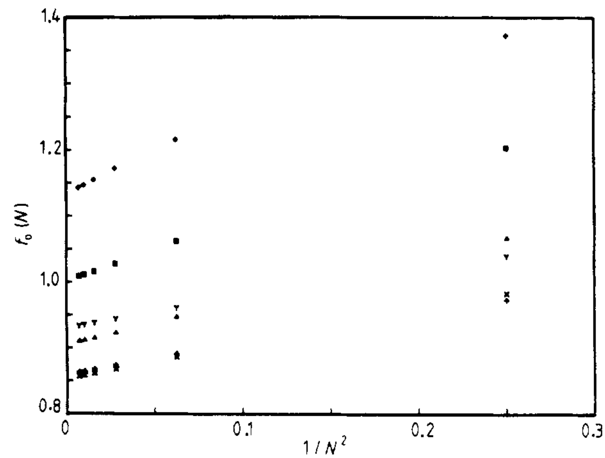

Figure 3. As in figure 2 for positive values of K_4 (◆, 1; ■, 0.8; ▲, 0.6; Y, 0; ×, 0.4; +, 0.2).

Extrapolated values A_0 and f^0 may then be obtained through a three-point fit (figure 4, tables 1 and 2). A_0 remains almost constant in the range $-1 \leqslant K_4 \leqslant 0.2$. The correction to scaling exponent y is then near to 2 and correctly related to the exponent for f^0, $y+2$. For larger K_4 values, either an anomalously large correction to scaling exponent is obtained, or the three-point fit is impossible. The strip widths are probably too small to observe the asymptotic regime. The poor convergence in this domain has already been observed for the exponent y_t in a phenomenological renormalisation group study (Nightingale 1977).

The value of A_0 obtained in the region $K_4 \leqslant 0.2$ where the extrapolation may be trusted is twice as large as the exactly known Ising result (Ferdinand and Fisher 1969)

$$A_0^{\text{Ising}} = \pi/12 \simeq 0.261\,799\,388. \tag{2.5}$$

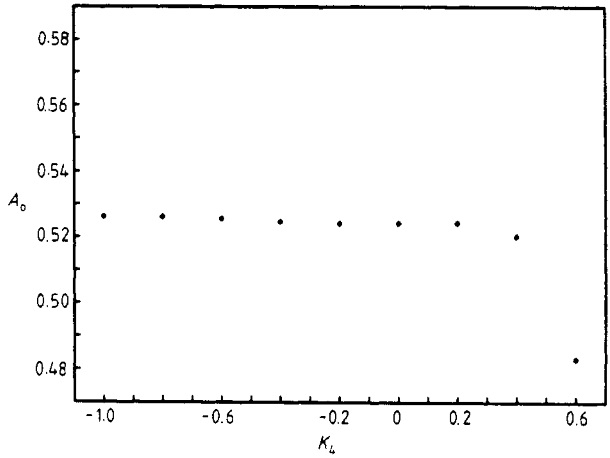

Figure 4. Extrapolated values of the free energy amplitude of the Baxter model as a function of the four-spin interaction K_4.

Table 1. Amplitude of the free energy density $A_0(N)$ on the largest strips as given by equations (2.3) in the text as a function of the four-spin interaction K_4. When possible extrapolated values A_0 are given. y is the correction to scaling exponent.

K_4	$A_0(7)$	$A_0(9)$	$A_0(11)$	A_0	y
−1.0	0.508 155 784	0.515 448 757	0.519 081 640	0.5261	2.1
−0.8	0.508 011 097	0.515 295 850	0.518 923 873	0.5260	2.1
−0.6	0.507 436 994	0.514 714 037	0.518 344 779	0.5254	2.1
−0.4	0.505 687 834	0.513 087 049	0.516 837 968	0.5245	2.0
−0.2	0.502 605 821	0.510 622 085	0.514 817 181	0.5241	1.9
0	0.500 881 372	0.509 463 824	0.513 986 590	0.5242	1.8
0.2	0.503 588 520	0.511 053 534	0.515 036 638	0.5244	1.8
0.4	0.523 721 172	0.521 064 172	0.521 002 422	0.5203	4.2
0.6	0.636 540 470	0.583 690 175	0.554 898 944	0.4831	1.7
0.8	0.957 443 966	0.850 464 061	0.745 170 525	—	—
1.0	1.423 036 464	1.404 280 123	1.291 074 492	—	—

Table 2. As in table 1 for the bulk free energy density $f^0(N)$ given by equation (2.4) in the text. f^0 is the extrapolated value and the correction to scaling exponent is $y+2$.

K_4	$f^0(7)$	$f^0(9)$	$f^0(11)$	f^0	$y+2$
−1.0	1.786 416 271	1.786 302 318	1.786 265 990	1.786 236	4.0
−0.8	1.590 146 690	1.590 032 866	1.589 996 586	1.589 967	4.0
−0.6	1.398 405 703	1.398 291 999	1.398 255 692	1.398 226	4.0
−0.4	1.216 485 497	1.216 369 885	1.216 332 376	1.216 301	3.9
−0.2	1.054 771 892	1.054 646 638	1.054 604 687	1.054 568	3.8
0	0.929 913 375	0.929 779 274	0.929 734 047	0.929 694	3.8
0.2	0.859 963 502	0.859 846 861	0.859 807 031	0.859 771	3.7
0.4	0.853 610 176	0.853 651 692	0.853 652 309	0.853 660	5.7
0.6	0.905 104 497	0.905 930 283	0.906 218 196	0.906 489	3.6
0.8	1.001 487 105	1.003 158 666	1.004 211 601	1.008 752	1.0
1.0	1.132 802 476	1.133 095 543	1.134 227 600	—	—

This discrepancy simply comes from a geometrical factor. When K_4 vanishes one gets two decoupled Ising sublattices. In order to get the free energy density on one of these, the Baxter free energy density must be halved.

3. Discussion

Our numerical results suggest that the free energy amplitude of the Baxter model is equal to $2A_0^{\text{Ising}}$, independent of K_4. We now show that this is true at least in the first order in K_4.

Consider the Baxter model on an $L\times\infty$ strip with periodic boundary conditions near to its decoupling point $K_4=0$. Using l as unit length, the free energy density is given by

$$f_0(L)=\lim_{M\to\infty}\frac{1}{NM}\ln \mathrm{Tr}\left(\exp(-\beta H_0)\prod_{(ijkl)}\exp K_4 s_i s_j s_k s_l\right) \tag{3.1}$$

where H_0 corresponds to the two decoupled Ising sublattices. A perturbation expansion gives

$$f_0(L)=2f_0^{\text{Ising}}(L)+K_4\langle s_i s_k\rangle_0^2+\mathrm{O}(K_4^2) \tag{3.2}$$

where $\varepsilon^{\text{Ising}}(K_2)=\langle s_i s_k\rangle_0$ is a first-neighbour Ising correlation function. One may notice that f_0^{Ising} is the Ising free energy density with l taken as unit length; since the unit surface is the half of the surface per spin, the Ising free energy per spin is twice this value. This is the reason why f^0 in table 1 converges towards the known Ising free energy per site on the square lattice (Ferdinand and Fisher 1969)

$$2G/\pi+\tfrac{1}{2}\ln 2\simeq 0.929\,695\,398 \tag{3.3}$$

where $G=$ Catalan's constant when K_4 vanishes.

To the first order in K_4, the correlation function in equation (3.2) is to be evaluated at the Ising critical point $K_c=1/2\ln(1+\sqrt{2})$. The square lattice Ising model remains self-dual on a strip with periodic boundary conditions and this property may be used to show that the internal energy density $u^{\text{Ising}}(K_c)$ and $\varepsilon^{\text{Ising}}(K_c)$ keep their bulk values (Syozi 1972, Debierre and Turban 1983). As a consequence the second term in equation (3.2) is size independent. $f_0^{\text{Ising}}(L)$ must be evaluated on the Baxter critical line $K_{2c}(K_4)$ such that

$$t=K_{2c}(K_4)-K_c=-K_4/\sqrt{2}+\mathrm{O}(K_4^2) \tag{3.4}$$

according to equation (1.4). One knows from finite-size scaling theory (Privman and Fisher 1984) that

$$f_0^{\text{Ising}}(L,t)=f^{0\,\text{Ising}}(t)+L^{-2}Y(c_1 tL^{y_t}) \tag{3.5}$$

where the first term gives the bulk free energy density. In the second $Y(x)$ is a universal function with $Y(0)=A_0^{\text{Ising}}$ and c_1 is a non-universal metric factor. Since we are looking for the size dependence we only need to consider the second term in equation (3.5). To the first order in K_4 one obtains

$$L^{-2}Y(c_1 tL^{y_t})=A_0^{\text{Ising}}L^{-2}-\frac{c_1K_4}{\sqrt{2}}L^{-2+y_t}\left.\frac{\mathrm{d}Y}{\mathrm{d}x}\right|_{x=0}. \tag{3.6}$$

The critical internal energy density on the strip is given by

$$u^{\text{Ising}}(K_c, L) = K_c \frac{\partial f_0^{\text{Ising}}}{\partial t}(L, t)\bigg|_{t=0}$$

$$= u^{\text{Ising}}(K_c) + c_1 K_c L^{-2+y_t} \frac{\mathrm{d}Y}{\mathrm{d}x}\bigg|_{x=0} \tag{3.7}$$

where the first term on the right gives the bulk value. Since, as mentioned above, the critical internal energy density on the strip is size independent, the amplitude $\mathrm{d}Y/\mathrm{d}x|_{x=0}$ must vanish. The second term in equation (3.6) vanishes too and to the first order in K_4 all the size dependence of the free energy amplitude in the Baxter model is provided by the Ising contribution

$$A_0 = 2A_0^{\text{Ising}} + \mathrm{O}(K_4^2). \tag{3.8}$$

Note added. After submission of this paper, we learned that, using conformal invariance, the free energy amplitude may be related to the central charge of the Virasoro algebra (Blöte *et al* 1986, Affleck 1986). The Baxter model has been studied in the first of these works and our suggestion that the Baxter free energy amplitude is twice the Ising one is confirmed by the exact result of Blöte *et al.*

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